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Multiply Universal Holomorphic Functions

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DEDICATED TO MY PARENTS

Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$, be an open set with simply connected components. In Theorem 1 we prove the existence of a holomorphic function ϕ on $\mathcal{O}$, which has together with all its derivatives and all its antiderivatives six universal properties at the same time (based on the behaviour of sequences of derivatives or antiderivatives, overconvergence-phenomena, or properties of translates). In Theorem 2 we show that the family of all functions with these universal properties is a dense subset of the metric space $H(\mathcal{O})$ of all holomorphic functions on $\mathcal{O}$, if $H(\mathcal{O})$ is endowed with the usual compact-open topology.

1. INTRODUCTION

Throughout this paper we assume that $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$, is an open set with simply connected components, i.e., there exists a finite or countable set I with $\mathcal{O} = \bigcup_{v \in I} G_v$, and the G_v are pairwise disjoint simply connected domains. We suppose that the function ϕ is holomorphic on $\mathcal{O}$ (which means that its restriction to any of the components G_v is holomorphic on G_v in the ordinary sense). By $H(\mathcal{O})$ we denote, as usual, the family of all functions which are holomorphic on $\mathcal{O}$.

If $\phi \in H(\mathcal{O})$ and $j \in \mathbb{N}_0$, we denote by $\phi^{(j)}$ the derivative of order j, and if $j \in \mathbb{N}$, we use the abbreviation $\phi^{(-j)}$ for an (arbitrary but fixed) antiderivative of order j for ϕ, i.e., we have

$$\frac{d^j}{dz^j}\,\phi^{(-j)}(z) = \phi(z) \qquad \text{for all} \quad z \in \mathcal{O}.$$

Such a function $\phi^{(-j)}$ is also called a j-fold antiderivative for ϕ. If $\phi_1^{(-j)}$ and $\phi_2^{(-j)}$ are both j-fold antiderivatives for ϕ, then we have

$$\phi_1^{(-j)}(z) - \phi_2^{(-j)}(z) = \chi(z),$$

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where χ is a function whose restriction to any of the components G_ν is a certain polynomial χ_ν of degree less than j.

A sequence $\{\phi^{(-j)}\}_{j\in\mathbb{N}}$ is called a "strict" sequence of antiderivatives, if the $\phi^{(-j)}$ are antiderivatives of order j for ϕ but satisfy in addition the stronger assumption

$$\frac{d}{dz}\phi^{(-j)}(z)=\phi^{(-j+1)}(z) \qquad \text{for all} \quad j\in\mathbb{N} \qquad \text{and all} \quad z\in\mathcal{O}.$$

In this paper we deal with the following problem. We fix a function $\phi\in H(\mathcal{O})$ (or a derivative or an antiderivative), and by applying simple analytic operations to this function we construct a sequence, which we associate with ϕ, and ask for the approximation properties of such a sequence: What functions on what subsets are obtainable as limits of such a sequence? We consider several possibilities to make these qualitative remarks more precise; for instance, we may study the following operations:

(a) We associate with the function ϕ its sequence $\{\phi^{(n)}\}$ of derivatives.

(b) We associate with the function ϕ a (strict) sequence $\{\phi^{(-n)}\}$ of antiderivatives.

(c) We expand the function ϕ in a power series around a point $z_0\in\mathcal{O}$ and associate with ϕ the sequence of partial sums of this power series.

(d) We associate with the function ϕ a sequence of "translates" $\{\phi(a_n z+b_n)\}$, where it is claimed that $\{b_n\}$ tends to a prescribed boundary point of $\mathcal{O}$, that $\{a_n\}$ tends to zero, and that $a_n z+b_n\in\mathcal{O}$ if z belongs to a specified subset of $\mathbb{C}$.

It is not immediately clear which approximation properties the sequences of type (d) have (where we deal with refinements of classical cluster sets). However, by carrying out operation (a), (b), or (c), the corresponding sequences only permit the approximation of very natural functions:

The sequence $\{\phi^{(n)}(z)\}$ of derivatives may diverge, but if it converges compactly on $\mathcal{O}$ then the limit function φ must satisfy $\varphi'(z)=\varphi(z)$ for all $z\in\mathcal{O}$ and hence we have $\varphi(z)=c_\nu e^z$ in each of the components G_ν of $\mathcal{O}$. The same holds for a strict sequence of antiderivatives. If we consider a power series $\phi(z)=\sum_{\nu=0}^{\infty}a_\nu(z-z_0)^\nu$, then its sequence of partial sums converges compactly to $\phi(z)$ in the greatest disk $D(z_0)$ around z_0 in which ϕ is holomorphic and diverges in any point of $\overline{D(z_0)}^c$.

But in any of these cases (a), (b), or (c) we may ask how the situation changes if we consider a subsequence instead of the total sequence. As a consequence of our Theorem 1 it will follow that the behaviour of such subsequences may be quite irregular.

2. STATEMENT OF THE MAIN RESULT

We start with some notations. By $\mathscr{M}$ we denote the family of all compact subsets $B \subset \mathbb{C}$ with connected complement. For any $B \in \mathscr{M}$ we denote by $A(B)$ the set of all functions which are continuous on B and holomorphic in the interior $\mathring{B}$ of B.

If a sequence $\{f_n\}$ of functions converges to the function f uniformly on a set S, then we write

$$f_n(z) \underset{S}{\Longrightarrow} f(z).$$

If S is an open set and if $\{f_n\}$ converges compactly to f on S, then we write

$$f_n(z) \underset{S}{\Rrightarrow} f(z).$$

The problems of the existence of so-called "universal functions" and their correspondence with the "universal approximation" of functions are classical. The first example is due to Birkhoff [1], who proved in 1929 the existence of a universal entire function ϕ with the property that for an arbitrary entire function f there exists a sequence $\{\zeta_n\}$ with $\zeta_n \to \infty$ and $\phi(z+\zeta_n) \underset{\mathbb{C}}{\Rrightarrow} f(z)$ for $n \to \infty$.

Since then many papers have dealt with this subject; the approximation theorems of Runge and Mergelyan are basic tools for the construction of functions which are universal in a certain specified sense (cf. [21], where a brief resumé of the history of this topic is given).

Several authors [5, 9–11,13] proved that—far from being a rare phenomenon—the spaces of certain universal functions are residual sets.

The first (and so far as we know, the only) example of a "multiply universal" function was given by Blair and Rubel [3]. The authors produced an entire function ϕ, whose sequence of derivatives $\{\phi^{(n)}\}$, a strict sequence $\{\phi^{(-n)}\}$ of antiderivatives, and a sequence of translates $\{\phi(z+\zeta_n)\}$ are dense in the space of all entire functions (endowed with the topology of compact convergence).

The main purpose of this paper is to prove the existence of multiply universal functions, which are holomorphic on $\mathcal{O}$ and have—together with all their derivatives and all antiderivatives—six universal properties at the same time. We shall also show (in Section 5) that the set $U(\mathcal{O})$ of all these multiply universal functions is a dense subset of the space $H(\mathcal{O})$, established with the topology of compact convergence.

Our main result is the following

THEOREM 1. *Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$, be an open set with simply connected components. Then there exists a function ϕ, which is holomorphic exactly on $\mathcal{O}$ and has the following properties*:

(A) *For all* $B \in \mathcal{M}$, $B \subset \mathcal{O}$, *and all* $f \in A(B)$ *there exists a sequence* $\{n_k\}$ *of natural numbers with*

$$\phi^{(n_k)}(z) \underset{B}{\Longrightarrow} f(z) \qquad (k \to \infty).$$

(B) *For all* $B \in \mathcal{M}$, $B \subset \mathcal{O}$, *and all* $f \in A(B)$ *there exists a suitable sequence* $\{\phi_{B,f}^{(-n)}\}_{n \in \mathbb{N}}$ *of n-fold antiderivatives* $\phi_{B,f}^{(-n)}$ *for* ϕ *(depending on* B *and* f*) such that*

$$\phi_{B,f}^{(-n)}(z) \underset{B}{\Longrightarrow} f(z) \qquad (n \to \infty).$$

(C) *There exists a strict sequence* $\{\phi^{(-n)}\}$ *of n-fold antiderivatives* $\phi^{(-n)}$ *for* ϕ, *such that for all* $B \in \mathcal{M}$, $B \subset \mathcal{O}$, *and all* $f \in A(B)$ *there exists a sequence* $\{n_k\}$ *of natural numbers with*

$$\phi^{(-n_k)}(z) \underset{B}{\Longrightarrow} f(z) \qquad (k \to \infty).$$

(Da) *There exists a sequence* $\{p_k\}$ *of natural numbers with the following property. Consider a derivative or any antiderivative* $\phi^{(j)}$ *and an arbitrary* $z_0 \in \mathcal{O}$. *Suppose that*

$$\phi^{(j)}(z) := \sum_{\nu=0}^{\infty} a_\nu^{(j, z_0)}(z - z_0)^\nu$$

is the power series expansion of $\phi^{(j)}$ *around* z_0; *then the sequence* $\{\sum_{\nu=0}^{p_k} a_\nu^{(j, z_0)}(z - z_0)^\nu\}$ *converges compactly on* $\mathcal{O}$. *The limit function is the derivative* $\phi^{(j)}$ *if* $j \in \mathbb{N}_0$ *and an antiderivative of order* $-j$ *if* $-j \in \mathbb{N}$.

(Db) *If* $\overline{\mathcal{O}}^c \neq \emptyset$, *then for all* $z_0 \in \mathcal{O}$, *all* $B \in \mathcal{M}$, $B \subset \overline{\mathcal{O}}^c$, *and all* $f \in A(B)$ *there exists a subsequence* $\{p_k^*\}$ *of* $\{p_k\}$ *with*

$$\sum_{\nu=0}^{p_k^*} a_\nu^{(j, z_0)}(z - z_0)^\nu \underset{B}{\Longrightarrow} f(z) \qquad (k \to \infty).$$

(E) *For all* $B \in \mathcal{M}$ *and all* $f \in A(B)$, *for any derivative and any antiderivative* $\phi^{(j)}$ *and for all* $\zeta \in \partial\mathcal{O}$ *there exist sequences* $\{a_n\}$ *and* $\{b_n\}$ *with* $a_n \to 0$, $b_n \to \zeta$, *such that* $a_n z + b_n \in \mathcal{O}$ *for all* $n \in \mathbb{N}$ *and all* $z \in B$ *with the property*

$$\phi^{(j)}(a_n z + b_n) \underset{B}{\Longrightarrow} f(z) \qquad (n \to \infty).$$

Methods of operator theory are often useful for establishing the existence of universal elements (see for example [9]). However, due to the generally complicated structure of disconnected open sets under consideration, it seems that these methods are not applicable in the present situation. So we will give an elementary proof of Theorem 1, which essentially uses the theorems of Runge and Mergelyan on complex approximation. Although only rudimentary methods are involved, we will admit that the proof may be considered as a "technical tour de force."

3. AUXILIARY RESULTS

For the proof of Theorem 1 two lemmas are needed.

LEMMA 1. *There exists an entire function φ with the following property. For all $B \in \mathscr{M}$ and all $f \in A(B)$ there exists a sequence $\{n_k\}$ of natural numbers with*

$$\varphi^{(n_k)}(z) \underset{B}{\Longrightarrow} f(z).$$

This result is essentially due to MacLane [15]; see also Blair and Rubel [2].

We say that a power series $\sum_{\nu=0}^{\infty} a_\nu (z-z_0)^\nu$ has Ostrowski gaps $\{p_k, q_k\}$ if p_k, q_k are natural numbers with the properties

$$p_1 < q_1 \leqslant p_2 < q_2 \leqslant \cdots, \qquad \lim_{k \to \infty} \frac{q_k}{p_k} = \infty$$

and

$$a_\nu = 0 \qquad \text{for} \quad \nu \in \bigcup_{k \in \mathbb{N}} (p_k, q_k).$$

LEMMA 2. *Let the function f be holomorphic in the domain G. Suppose that the power series of f around a point $z_0 \in G$,*

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu (z-z_0)^\nu,$$

possesses Ostrowski gaps $\{p_k, q_k\}$. Let be given any point $w_0 \in G$ and consider the power series expansion of f around w_0:

$$f(z) = \sum_{\nu=0}^{\infty} b_\nu (z-w_0)^\nu,$$

Then we have

$$\sum_{\nu=0}^{p_k} a_\nu (z-z_0)^\nu - \sum_{\nu=0}^{p_k} b_\nu (z-w_0)^\nu \underset{\mathbb{C}}{\Longrightarrow} 0.$$

For a proof see Luh [16, Theorem 1].

4. PROOF OF THEOREM 1.

1. (a) Let $I = \{1, 2, ...\}$ be a finite or countable set with $\mathcal{O} = \bigcup_{\nu \in I} G_\nu$, where the G_ν are pairwise disjoint, simply connected domains (the components of $\mathcal{O}$). For each $\nu \in I$ we choose a sequence $\{G_{\nu, n}\}_{n \in \mathbb{N}}$ of Jordan domains $G_{\nu, n}$ having rectifiable boundaries $\partial G_{\nu, n}$, with

$$\overline{G_{\nu, n}} \subset G_{\nu, n+1} \subset G_\nu \qquad \text{for all} \quad n \in \mathbb{N},$$

and that for any compact set $K \subset G_\nu$ there exists an $n_0 \in \mathbb{N}$ such that $K \subset G_{\nu, n_0}$. We suppose that $I_n := I \cap \{1, 2, ..., n\}$ and consider the open set

$$\mathcal{O}_n := \bigcup_{\nu \in I_n} G_{\nu, n}.$$

For any $\nu \in I$ we fix a point $z_\nu^* \in G_{\nu, 1}$ and define

$$\Delta_\nu := \operatorname{dist}(z_\nu^*, \partial G_\nu).$$

If $I = \{1, ..., N\}$ is a finite set, let $z_n^* := z_N^*$ for all $n \geqslant N$.

We consider the polynomials

$$\Omega_0(z) \equiv 0; \qquad \Omega_n(z) = \prod_{\nu=1}^{n} (z - z_\nu^*) \qquad \text{if} \quad n \in \mathbb{N}.$$

(b) Suppose that $\{\zeta_{\nu, k}\}_{k \in \mathbb{N}}$ is a sequence of points which is dense in ∂G_ν. For fixed $k \in \mathbb{N}$, $j \in \mathbb{Z}$, $\nu \in I$ we choose sequences $\{z_{\nu, k, j, n}\}_{n \in \mathbb{N}}$ of pairwise different points with the properties that for fixed $\nu \in I$

$$\lim_{n \to \infty} z_{\nu, k, j, n} = \zeta_{\nu, k} \qquad \text{for all} \quad j \in \mathbb{Z}$$

and

$$z_{\nu, k, j, n} \in G_{\nu, n+1} \setminus \overline{G_{\nu, n}} \qquad \text{for} \quad k = 1, ..., n \qquad \text{and} \qquad j = 0, \pm 1, ..., \pm n.$$

Next we choose $0 < \delta_n < 1/n$ so small that the closed circles

$$D_{\nu, k, j, n} := \{z : |z - z_{\nu, k, j, n}| \leqslant \delta_n\}$$

are pairwise disjoint for $\nu \in I_n$; $k = 1, ..., n$; $j = 0, \pm 1, ..., \pm n$, and that

$$D_{\nu, n} := \bigcup_{k=1}^{n} \bigcup_{|j| \leqslant n} D_{\nu, k, j, n} \subset G_{\nu, n+1} \setminus \overline{G_{\nu, n}}$$

holds. Then we have

$$D_n := \bigcup_{\nu \in I_n} D_{\nu, n} \subset \mathcal{O}_{n+1} \setminus \overline{\mathcal{O}}_n,$$

and D_n is a compact set with connected complement.

(c) The set $\overline{\mathcal{O}}^c = \mathbb{C} \setminus \mathcal{O}$ is open, and if it is not the empty set then there exists a finite or countable set $J = \{1, 2, ...\}$ such that $\overline{\mathcal{O}}^c = \bigcup_{\nu \in J} U_\nu$, where the U_ν are pairwise disjoint domains. We define $J_n := J \cap \{1, 2, ..., n\}$. For any $\nu \in J$ we denote by $\mathcal{P}_\nu$ the family of all Jordan domains which are contained in U_ν and are bounded by a closed polygon with vertices in U_ν and which have rational real and imaginary parts. This family $\mathcal{P}_\nu$ is countable; let $\{H_{\nu, n}\}_{n \in \mathbb{N}}$ be an enumeration of $\mathcal{P}_\nu$.

For fixed $n \in \mathbb{N}$ we consider the collection $\mathcal{H}_n$ of all open sets of the type

$$\bigcup_{\nu \in J_n} H_{\nu, \mu_\nu} \qquad \text{where} \quad 1 \leqslant \mu_\nu \leqslant n.$$

The family $\mathcal{H}$ of all these $\mathcal{H}_n$ is again countable; let $\{H_k\}_{k \in \mathbb{N}}$ be an enumeration of $\mathcal{H}$. Any such open set H_k consists of a finite number of pairwise disjoint Jordan domains.

Now let $\{Q_n\}_{n \in \mathbb{N}}$ be an enumeration of all polynomials with coefficients, whose real and imaginary parts are rational.

Any $n \in \mathbb{N}$ has a unique representation of the type

$$n = \binom{m}{2} + \mu \qquad \text{where} \quad m \in \mathbb{N}, \quad 1 \leqslant \mu \leqslant m.$$

We define

$$J_n = J_{\binom{m}{2} + \mu} := H_\mu,$$

$$T_n(z) = T_{\binom{m}{2} + \mu}(z) := Q_{m - \mu + 1}(z).$$

For any J_n we have $\overline{J}_n \subset \overline{\mathcal{O}}^c$ and $J_n = \bigcup_{\ell=1}^{N_n} h_{\ell, n}$ with Jordan domains $h_{\ell, n}$ (bounded by polygons), which are contained in different components of $\overline{\mathcal{O}}^c$. For any of these $h_{\ell, n}$ we choose a Jordan domain $h^*_{\ell, n}$ with rectifiable boundary $\partial h^*_{\ell, n}$, such that $h_{\ell, n}$ and $h^*_{\ell, n}$ are contained in the same component of $\overline{\mathcal{O}}^c$, that $\overline{h_{\ell, n}} \subset \overline{h^*_{\ell, n}}$ and $\operatorname{dist}(\overline{h_{\ell, n}}, \partial h^*_{\ell, n}) < 1/n$ hold. We define

$$J^*_n := \bigcup_{\ell=1}^{N_n} h^*_{\ell, n}.$$

2. According to Lemma 1 there exists an entire function φ, such that the sequence of derivatives $\{\varphi^{(n)}\}$ is for all sets $B \in \mathscr{M}$ dense in the space $A(B)$.

(a) By an inductive procedure we shall construct a sequence $\{P_\mu\}_{\mu \in \mathbb{N}_0}$ of polynomials of the type

$$P_\mu(w) = \Pi_\mu(w) \cdot \{\Omega_\mu(w)\}^{\lambda_\mu}. \tag{1}$$

To this end we note that any natural number μ has a unique representation of the form

$$\mu = n^2 + n + m = =: (n, m) \qquad \text{where} \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad |m| \leqslant n.$$

We start our induction by setting

$$\Pi_0(w) \equiv 1, \qquad \lambda_0 = 1, \qquad s_0 = 1.$$

Let be given an $n \in \mathbb{N}$ and an $m \in \mathbb{Z}$ with $|m| \leqslant n$; we abbreviate $\mu := (n, m)$ and suppose that the natural numbers

$$\lambda_0, ..., \lambda_{\mu-1}; \qquad s_0, ..., s_{\mu-1}$$

and the polynomials

$$\Pi_0, ..., \Pi_{\mu-1}$$

have already been constructed. By (1) the polynomials $P_0, ..., P_{\mu-1}$ also are well defined. We further assume that for $\kappa = 0, ..., \mu-1$ a sequence $\{P_\kappa^{(j)}\}_{-j \in \mathbb{N}}$ of strict antiderivatives for P_κ is determined.

We denote by $\rho_{\mu-1}$ the degree of the polynomial $\Pi_{\mu-1}$ and choose the natural number $s_\mu > s_{\mu-1}$ so great that the properties

$$s_\mu > (\mu-1)\lambda_{\mu-1} + \rho_{\mu-1}$$

and

$$\max_{\overline{\mathcal{O}_\mu}} |\varphi^{(s_\mu)}(w) - Q_\mu(w)| < \frac{1}{2^\mu} \tag{2}$$

hold (which is possible by Lemma 1). Next we choose a natural number λ_μ with

$$\lambda_\mu > (\mu-1)\{(\mu-1)\,\lambda_{\mu-1} + \rho_{\mu-1} + \mu - 1\}. \tag{3}$$

We consider a function F_μ, which satisfies for $\nu \in I_\mu$; $k=1, ..., \mu$; $j=0, \pm 1, ..., \pm\mu$; and all $w \in D_{\nu,k,j,\mu}$ the condition

$$\frac{d^{\mu+j}}{dw^{\mu+j}} F_\mu(w) = Q_\mu\left(\frac{2\mu}{\mu+1} \cdot \frac{\Delta\nu}{\delta_\mu} \cdot (w - z_{\nu,k,j,\mu}) + z_\nu^*\right) - \sum_{\kappa=0}^{\mu-1} P_\kappa^{(j)}(w),$$

and let F_μ^* be any antiderivative of order $\mu + m = (n, m) + m$ for the polynomial

$$T_n(w) = \sum_{\kappa=0}^{\mu-1} P_\kappa^{(m)}(w).$$

According to Runge's approximation theorem there exists a polynomial $r_\mu(w) \not\equiv 0$, which satisfies with suitable positive constants $\varepsilon_\mu, \varepsilon'_\mu, \varepsilon''_\mu$ the following conditions simultaneously:

$$\max_{\overline{\mathcal{O}_\mu}} |r_\mu(w)| < \varepsilon_\mu \cdot \{\max_{\overline{\mathcal{O}_\mu}} |\Omega_\mu(w)|\}^{-\lambda_\mu - 2\mu}, \tag{4}$$

$$\max_{D_\mu} \left| r_\mu(w) - \frac{F_\mu(w)}{\{\Omega_\mu(w)\}^{\lambda_\mu + 2\mu}} \right| < \varepsilon'_\mu \cdot \{\max_{D_\mu} |\Omega_\mu(w)|\}^{-\lambda_\mu - 2\mu}, \tag{5}$$

and if $\overline{\mathcal{O}}^c \neq \varnothing$ we additionally claim

$$\max_{\overline{J_n^*}} \left| r_\mu(w) - \frac{F_\mu^*(w)}{\{\Omega_\mu(w)\}^{\lambda_\mu + 2\mu}} \right| < \varepsilon''_\mu \cdot \{\max_{\overline{J_n^*}} |\Omega_\mu(w)|\}^{-\lambda_\mu - 2\mu}. \tag{6}$$

If for $n \in \mathbb{N}$ the integer m runs from $-n$ to n we obtain the polynomials $r_{n^2}, ..., r_{(n+1)^2-1}$ and hence by induction we get the sequence $\{r_\mu(w)\}_{\mu \in \mathbb{N}_0}$ of polynomials.

(b) For $\mu \in \mathbb{N}$ we use the abbreviations

$$R_\mu(w) := r_\mu(w) \cdot \{\Omega_\mu(w)\}^{\lambda_\mu + 2\mu}; \qquad P_\mu(w) := \frac{d^\mu}{dw^\mu} R_\mu(w)$$

and obtain with a well-defined polynomial Π_μ,

$$P_\mu(w) = \Pi_\mu(w) \cdot \{\Omega_\mu(w)\}^{\lambda_\mu}.$$

For $-\mu \leqslant j \in \mathbb{Z}$ we define

$$P_\mu^{(j)}(w) = \frac{d^{\mu+j}}{dw^{\mu+j}} R_\mu(w)$$

and for $j < -\mu$ we choose $P_\mu^{(j)}$ so that

$$\frac{d}{dw} P_\mu^{(j)}(w) = P_\mu^{(j+1)}(w)$$

holds. We thereby have determined a strict sequence of antiderivatives of the polynomial P_μ.

3. We investigate some properties of the polynomials P_μ.

(a) From (4) we obtain for all $\mu > 1$

$$\max_{\overline{\mathcal{O}_\mu}} |R_\mu(w)| < \varepsilon_\mu. \tag{7}$$

Suppose that $\nu \in I_\mu$ and that $\overline{G_{\nu,\mu-1}}$ is any of the components of $\overline{\mathcal{O}_{\mu-1}}$, then for $-\mu \leqslant j \leqslant s_\mu$ and all $z \in \overline{G_{\nu,\mu-1}}$ we have

$$P_\mu^{(j)}(z) = \frac{(\mu+j)!}{2\pi i} \int_{\partial G_{\nu,\mu}} \frac{R_\mu(w)}{(w-z)^{\mu+j+1}}\, dw.$$

Estimating this integral in a straightforward way using (7), and then taking the maximum for $\nu \in I_{\mu-1}$ and $-\mu \leqslant j \leqslant s_\mu$, we get

$$\max_{\overline{G_{\nu,\mu-1}}} |P_\mu^{(j)}(z)| < \frac{1}{2^\mu},$$

if ε_μ has been chosen sufficiently small. Since $G_{\nu,\mu-1}$ was an arbitrary component of $\mathcal{O}_{\mu-1}$ we have

$$\max_{\overline{\mathcal{O}_{\mu-1}}} |P_\mu^{(j)}(z)| < \frac{1}{2^\mu} \qquad \text{for all} \quad j \qquad \text{with} \quad -\mu \leqslant j \leqslant s_\mu. \tag{8}$$

(b) In a similar way we obtain

$$\max_{D_{\mu-1}} |P_\mu^{(j)}(z)| < \frac{1}{2^\mu} \qquad \text{for all} \quad j \qquad \text{with} \quad \mu > |j|. \tag{9}$$

(c) From (5) we get for $\mu \in \mathbb{N}$

$$\max_{D_\mu} |R_\mu(w) - F_\mu(w)| < \varepsilon'_\mu. \tag{10}$$

For $v \in I_\mu$; $k = 1, ..., \mu$; $j = 0, \pm 1, ..., \pm \mu$; and all z with $|z - z_{v,k,j,\mu}| \leqslant \delta_\mu/2$ we have

$$\sum_{\kappa=0}^{\mu} P_k^{(j)}(z) - Q_\mu\left(\frac{2\mu}{\mu+1}\cdot\frac{\Delta_v}{\delta_\mu}\cdot(z - z_{v,k,j,\mu}) + z_v^*\right)$$
$$= \frac{(\mu+j)!}{2\pi i}\int_{\partial D_{v,k,j,\mu}} \frac{R_\mu(w) - F_\mu(w)}{(w-z)^{\mu+j+1}}\,dw.$$

Estimating this integral using (10) and then taking the maximum for the values of v and j under consideration, we get

$$\max_{|z - z_{v,k,j,\mu}| \leqslant \delta_\mu/2}\left|\sum_{\kappa=0}^{\mu} P_\kappa^{(j)}(z) - Q_\mu\left(\frac{2\mu}{\mu+1}\cdot\frac{\Delta_v}{\delta_\mu}\cdot(z - z_{v,k,j,\mu}) + z_v^*\right)\right| < \frac{1}{2^\mu}, \quad (11)$$

if ε'_μ has been chosen sufficiently small.

(d) We suppose that $\overline{\mathcal{O}}^c \neq \emptyset$. From (6) we obtain for $\mu = (n, m) = n^2 + n + m$ with $n \in \mathbb{N}$ and $-m \leqslant n \leqslant m$

$$\max_{\overline{J_n^*}} |R_\mu(w) - F_\mu^*(w)| < \varepsilon''_\mu. \quad (12)$$

Let $h_{\ell,n}$ be one of the components of J_n, then we have for all $z \in \overline{h_{\ell,n}}$

$$\sum_{\kappa=0}^{\mu} P_\kappa^{(m)}(z) - T_n(z) = \frac{(\mu+m)!}{2\pi i}\int_{\partial h_{\ell,n}^*} \frac{R_\mu(w) - F_\mu^*(w)}{(w-z)^{\mu+m+1}}\,dw.$$

We estimate this integral by use of (12) and since $n^2 \leqslant \mu \leqslant n^2 + 2n$ and $\ell \leqslant N_n$, we get

$$\max_{\overline{h_{\ell,n}}}\left|\sum_{\kappa=0}^{\mu} P_\kappa^{(m)}(z) - T_n(z)\right| < \frac{1}{n},$$

if ε''_μ has been chosen sufficiently small. Since $h_{\ell,n}$ was an arbitrary component of J_n, it follows for fixed $m \in \mathbb{Z}$ and all $n \geqslant |m|$

$$\max_{\overline{J_n}}\left|\sum_{\kappa=0}^{n^2+n+m} P_\kappa^{(m)}(z) - T_n(z)\right| < \frac{1}{n}. \quad (13)$$

4. Let us now consider the polynomial series $\sum_{\mu=0}^{\infty} P_\mu(z)$.

Suppose that B is an arbitrary compact subset of $\mathcal{O}$, then there exists an $n_B \geqslant 2$ such that $B \subset \mathcal{O}_{\mu-1}$ for all $\mu > n_B$. If j is a fixed integer, (8) yields for all $\mu > n_B + |j|$

$$\max_B |P_\mu^{(j)}(z)| \leqslant \max_{\overline{\mathcal{O}_{\mu-1}}} |P_\mu^{(j)}(z)| < \frac{1}{2^\mu}.$$

Since B was arbitrary, it follows that the series $\sum_{\mu=0}^{\infty} P_{\mu}^{(j)}(z)$ converges compactly on $\mathcal{O}$ for any $j \in \mathbb{Z}$. Therefore the functions

$$\psi(z) := \sum_{\mu=0}^{\infty} P_{\mu}(z), \qquad \psi_0^{(j)}(z) := \sum_{\mu=0}^{\infty} P_{\mu}^{(j)}(z)$$

are holomorphic on $\mathcal{O}$. We define the function ϕ by

$$\phi(z) := \varphi(z) + \psi(z).$$

5. We consider any compact set $B \in \mathcal{M}$ with $B \subset \mathcal{O}$ and any function $f \in A(B)$.

(a) According to the theorem of Mergelyan we can choose a sequence $\{n_k\}$ with $n_k \to \infty$ and

$$\max_B |f(z) - Q_{n_k}(z)| < \frac{1}{k}.$$

There exists an $n_B \geqslant 2$ with $B \subset \mathcal{O}_{n-1}$ for all $n > n_B$. We suppose that $n > n_B$. It follows from (2) that

$$\max_B |\varphi^{(s_n)}(z) - Q_n(z)| \leqslant \max_{\overline{\mathcal{O}_n}} |\varphi^{(s_n)}(z) - Q_n(z)| < \frac{1}{2^n}.$$

For $0 \leqslant \mu < n$ we have $s_n > \mu\lambda_\mu + \rho_\mu$ and hence we obtain

$$\begin{aligned}\phi^{(s_n)}(z) &= \varphi^{(s_n)}(z) + \sum_{\mu=0}^{n-1} P_{\mu}^{(s_n)}(z) + \sum_{\mu=n}^{\infty} P_{\mu}^{(s_n)}(z)\\ &= \varphi^{(s_n)}(z) + \sum_{\mu=n}^{\infty} P_{\mu}^{(s_n)}(z).\end{aligned}$$

Since $s_n \leqslant s_\mu$ for all $\mu \geqslant n$ we get from (8)

$$\max_B |\phi^{(s_n)}(z) - \varphi^{(s_n)}(z)| \leqslant \sum_{\mu=n}^{\infty} \max_{\overline{\mathcal{O}_{\mu-1}}} |P_{\mu}^{(s_n)}(z)| < \frac{1}{2^{n-1}}.$$

We therefore have $\phi^{(s_{n_k})}(z) \underset{B}{\Longrightarrow} f(z)$ which proves assertion (A).

(b) By [19, Theorem 3] there exists a sequence $\{\phi_{B,f}^{(-n)}\}_{n \in \mathbb{N}}$ of n-fold antiderivatives (depending on B and f), such that

$$\phi_{B,f}^{(-n)} \underset{B}{\Longrightarrow} f(z) \qquad (n \to \infty).$$

This proves assertion (B).

(c) By [20, Theorem] there exists a (universal) strict sequence $\{\phi^{(-n)}\}_{n\in\mathbb{N}}$ of n-fold antiderivatives, such that a subsequence $\{n_k\}$ corresponds with the given B and f, satisfying

$$\phi^{(-n_k)}(z) \underset{B}{\Longrightarrow} f(z) \qquad (k\to\infty).$$

This proves assertion (C).

6. Let any $j\in\mathbb{Z}$ be given and consider the derivative $\phi^{(j)}$ if $j\geqslant 0$ or any antiderivative $\phi^{(j)}$ of order $|j|$ if $j<0$.

Suppose that $\varphi_0^{(j)}$ is the derivative or an arbitrary (but fixed) antiderivative of order $|j|$ for φ. Then the function $\phi^{(j)}$ may be represented in the form

$$\phi^{(j)}(z)=\varphi_0^{(j)}(z)+\psi_0^{(j)}(z)+\chi_j(z), \tag{14}$$

where χ_j is a function whose restriction to a component G_ν of $\mathcal{O}$ is equal to a polynomial $\chi_{j,\nu}$ of degree less than $|j|$. Observe that $\varphi_0^{(j)}$ is an entire function for all $j\in\mathbb{Z}$ and that $\chi_j(z)\equiv 0$ if $j\geqslant 0$.

Let also any $z_0\in\mathcal{O}$ be given and consider the power series expansion of $\phi^{(j)}$ around z_0:

$$\phi^{(j)}(z)=\sum_{\nu=0}^{\infty} a_\nu^{(j,z_0)}(z-z_0)^\nu. \tag{15}$$

We shall prove the desired overconvergence properties of this series.

(a) Suppose that G_{m_0} is the component of $\mathcal{O}$ with $z_0\in G_{m_0}$. We consider the power series expansions of $\psi_0^{(j)}$ around z_0 and $z^*_{m_0}$:

$$\psi_0^{(j)}(z)=\sum_{\nu=0}^{\infty} c_\nu^{(j,z_0)}(z-z_0)^\nu, \tag{16}$$

$$\psi_0^{(j)}(z)=\sum_{\nu=0}^{\infty} c_\nu^{(j,z^*_{m0})}(z-z^*_{m_0})^\nu. \tag{17}$$

For $\mu>|j|$ we have

$$P_\mu^{(j)}(z)=\frac{d^{\mu+j}}{dz^{\mu+j}}R_\mu(z)=\frac{d^{\mu+j}}{dz^{\mu+j}}\left(r_\mu(z)\cdot\{\Omega_\mu(z)\}^{\lambda_\mu+2\mu}\right)=\Pi_{j,\mu}(z)\cdot\{\Omega_\mu(z)\}^{\lambda_\mu},$$

with a suitable polynomial $\Pi_{j,\mu}$ of degree $\rho_\mu-j\geqslant 0$. For $\mu>|j|+m_0$ we have

$$P_\mu^{(j)}(z)=(z-z^*_{m_0})^{\lambda_\mu}\cdot\prod_{\substack{\nu=1\\ \nu\neq m_0}}^{\mu}(z-z^*_\nu)^{\lambda_\mu}\cdot\Pi_{j,\mu}(z),$$

therefore the highest power of $(z-z^*_{m_0})$ in the polynomial $P^{(j)}_\mu$ (by its expansion around $z^*_{m_0}$) has an exponent not greater than $\mu\lambda_\mu+\rho_\mu-j$, while the least power of $(z-z^*_{m_0})$ in $P^{(j)}_{\mu+1}$ has an exponent at least $\lambda_{\mu+1}$. By (3) we have

$$\lambda_{\mu+1}>\mu\{\mu\lambda_\mu+\rho_\mu+\mu\}>\mu\lambda_\mu+\rho_\mu-j,$$

and it follows that the power series (17) is obtained (after a starting partial sum) by writing consecutively the terms of the series $\sum_{\mu=0}^\infty P^{(j)}_\mu(z)$ (by its expansion around $z^*_{m_0}$). If we define

$$p_k:=k\lambda_k+\rho_k+k, \qquad q_k:=\lambda_{k+1},$$

we obtain for sufficiently large k

$$\sum_{\nu=0}^{p_k} c_\nu^{(j,\, z^*_{m_0})}(z-z^*_{m_0})^\nu=\sum_{\mu=0}^{k} P^{(j)}_\mu(z). \tag{18}$$

And therefore for $k\to\infty$ the partial sums on the left-hand side converge to $\psi^{(j)}_0(z)$ compactly on $\mathcal{O}$. The power series (17) has Ostrowski gaps $\{p_k, q_k\}$, which satisfy by (3) the condition $q_k/p_k\geqslant k$, and it follows from Lemma 2 that

$$\sum_{\nu=0}^{p_k} c_\nu^{(j,\, z^*_{m_0})}(z-z^*_{m_0})^\nu=\sum_{\nu=0}^{p_k} c_\nu^{(j,\, z_0)}(z-z_0)^\nu \underset{\mathbb{C}}{\Longrightarrow} 0. \tag{19}$$

By (14) and (15) it follows immediately that

$$\sum_{\nu=0}^{p_k} a_\nu^{(j,\, z_0)}(z-z_0)^\nu \underset{\mathcal{O}}{\Longrightarrow} \phi^{(j)}(z)+\chi_{j,\,m_0}(z)-\chi_j(z) \qquad (k\to\infty),$$

which proves assertion (Da).

7. We assume that $\overline{\mathcal{O}}^c\neq\emptyset$ and that any compact set $B\in\mathcal{M}$ with $B\subset\overline{\mathcal{O}}^c$, any function $f\in A(B)$, a derivative or an antiderivative $\phi^{(j)}$, and a point $z_0\in\mathcal{O}$ are given. We again consider the power series (15).

If G_{m_0} is the component of $\mathcal{O}$ with $z_0\in G_{m_0}$, then we have by (14) for all $z\in G_{m_0}$

$$\phi^{(j)}(z)=\varphi^{(j)}_0(z)+\psi^{(j)}_0(z)+\chi_{j,\,m_0}(z).$$

We abbreviate $\tilde{f}(z):=f(z)-\varphi^{(j)}_0(z)-\chi_{j,\,m_0}(z)$. By Mergelyan's theorem we can choose a sequence $\{m_k\}$ with $m_k\geqslant k$ and

$$\max_B |\tilde{f}(z)-Q_{m_k}(z)|<\frac{1}{k}.$$

There exists an N with $B \subset \bigcup_{\nu \in J_N} U_\nu$, and the sets $B_\nu := B \cap U_\nu$ are empty or are compact sets in $\mathscr{M}$. It is not hard to show that (if N is chosen sufficiently large) for each $\nu \in J_N$ there exists a domain H_{ν, μ_ν} with $B_\nu \subset H_{\nu, \mu_\nu}$, i.e., we obtain

$$B = \bigcup_{\nu \in J_N} B_\nu \subset \bigcup_{\nu \in J_N} H_{\nu, \mu_\nu}$$

and therefore $B \subset H_{\ell_0}$ for a suitable ℓ_0. We define

$$n_k := \binom{m_k + \ell_0 - 1}{2} + \ell_0$$

and get

$$J_{n_k} = H_{\ell_0} \qquad \text{and} \qquad T_{n_k}(z) = Q_{m_k}(z).$$

According to (13), we obtain for all k with $n_k > |j|$,

$$\max_{\overline{J_{n_k}}} \left| \sum_{\mu=0}^{n_k^2 + n_k + j} P_\mu^{(j)}(z) - T_{n_k}(z) \right| < \frac{1}{n_k}.$$

If we define $p_k^* = p_{n_k^2 + n_k + j}$, then we get from (18)

$$\max_{\overline{H_{\ell_0}}} \left| \sum_{\nu=0}^{p_k^*} c_\nu^{(j, z_{m0}^*)} (z - z_{m0}^*)^\nu - Q_{m_k}(z) \right| < \frac{1}{n_k}.$$

By Lemma 2 we conclude

$$\sum_{\nu=0}^{p_k^*} c_\nu^{(j, z_0)} (z - z_0)^\nu \underset{B}{\Longrightarrow} \tilde{f}(z)$$

and therefore

$$\sum_{\nu=0}^{p_k^*} a_\nu^{(j, z_0)} (z - z_0)^\nu \underset{B}{\Longrightarrow} f(z).$$

This proves assertion (Db).

8. We study the translation properties of $\psi_0^{(j)}$.

(a) For fixed $\nu \in I, k \in \mathbb{N}$, and $n > \max\{\nu, k, |j|\}$ we have for all $w \in G_\nu$

$$\psi_0^{(j)}(w) - Q_n\left(\frac{2n}{n+1} \cdot \frac{\Delta_\nu}{\delta_n} \cdot (w - z_{\nu, k, j, n}) + z_\nu^*\right)$$
$$= \sum_{\mu=0}^{n} P_\mu^{(j)}(w) - Q_n\left(\frac{2n}{n+1} \cdot \frac{\Delta_\nu}{\delta_n} \cdot (w - z_{\nu, k, j, n}) + z_\nu^*\right) + \sum_{\mu=n+1}^{\infty} P_\mu^{(j)}(w).$$

By (9) we get

$$\max_{|w - z_{v,k,j,n}| \leqslant \delta_n/2} \left| \sum_{\mu=n+1}^{\infty} P_\mu^{(j)}(w) \right| \leqslant \sum_{\mu=n+1}^{\infty} \max_{D_{\mu-1}} |P_\mu^{(j)}(w)| < \frac{1}{2^n},$$

and together with (11) we obtain

$$\max_{|w - z_{v,k,j,n}| \leqslant \delta_n/2} \left| \psi_0^{(j)}(w) - Q_n \left(\frac{2n}{n+1} \cdot \frac{\Delta_v}{\delta_v} \cdot (w - z_{v,k,j,n}) + z_v^* \right) \right| < \frac{1}{2^{n-1}},$$

or equivalently

$$\max_{|z - z_v^*| \leqslant (n/(n+1))\, \Delta_v} \left| \psi_0^{(j)} \left(\frac{n+1}{2n} \cdot \frac{\delta_n}{\Delta_v} \cdot (z - z_v^*) + z_{v,k,j,n} \right) - Q_n(z) \right| < \frac{1}{2^{n-1}}. \quad (20)$$

(b) Suppose now that any $\zeta \in \partial G_v$, a compact set $B \in \mathscr{M}$, and a function $f \in A(B)$ are given. We choose an $R > 0$ such that

$$\tilde{B} := \left\{ z : z = \frac{w}{R} + z_v^*,\ w \in B \right\} \subset \{ z : |z - z_v^*| < \Delta_v \}$$

and consider the function $\tilde{f}(z) := f(R(z - z_v^*))$. Then we have $\tilde{f} \in A(\tilde{B})$ and by Mergelyan's theorem we can find an increasing sequence $\{n_m\}$ with

$$\max_{\tilde{B}} |\tilde{f}(z) - Q_{n_m}(z)| < \frac{1}{m}$$

and

$$\tilde{B} \subset \left\{ z : |z - z_v^*| \leqslant \frac{n_m}{n_m + 1} \Delta_v \right\}.$$

Hence we obtain from (20)

$$\max_{\tilde{B}} \left| \psi_0^{(j)} \left(\frac{n_m+1}{2n_m} \cdot \frac{\delta_{n_m}}{\Delta_v} \cdot (z - z_v^*) + z_{v,k,j,n_m} \right) - \tilde{f}(z) \right| < \frac{1}{2^{n_m} - 1} + \frac{1}{m} \leqslant \frac{2}{m}. \quad (21)$$

The point ζ is a limit point of the set of all points z_{v,k,j,n_m} and therefore we can find subsequences $\{k_s\}$ and $\{m_s\}$ such that

$$b_s := z_{v,k_s,j,n_{m_s}} \to \zeta \qquad \text{for} \quad s \to \infty.$$

We obtain

$$a_s := \frac{n_{m_s}+1}{2n_{m_s}} \cdot \frac{\delta n_{m_s}}{\Delta_\nu} \cdot \frac{1}{R} \to 0 \qquad \text{for} \quad s \to \infty,$$

and we have $a_s z + b_s \in G_\nu$ for all $s \in \mathbb{N}$ and all $z \in B$. It follows from (21) that

$$\max_B |\psi_0^{(j)}(a_s z + b_s) - f(z)| < \frac{2}{m_s}$$

and therefore $\psi_0^{(j)}(a_s z + b_s) \underset{B}{\rightrightarrows} f(z)$ for $s \to \infty$.

9. We finally study the translation properties of $\phi^{(j)}$.

Let a derivative or any antiderivative $\phi^{(j)}$, any boundary point $\zeta \in \partial\mathcal{O}$, a compact set $B \in \mathcal{M}$, and a function $f \in A(B)$ be given.

(a) We first assume that there exists an $\nu_0 \in I$ such that $\zeta \in \partial G_{\nu_0} \cap \mathbb{C}$. By (14) we have for all $z \in G_{\nu_0}$

$$\phi^{(j)}(z) = \varphi_0^{(j)}(z) + \psi_0^{(j)}(z) + \chi_{j,\nu_0}(z),$$

where $\varphi_0^{(j)}$ is an entire function and χ_{j,ν_0} is a polynomial. According to the translation properties of $\psi_0^{(j)}$ there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with

$$\begin{aligned} &\alpha_n \to 0, \quad \beta_n \to \zeta && \text{for } n \to \infty, \\ &\alpha_n z + \beta_n \in G_{\nu_0} && \text{for all } n \in \mathbb{N} \text{ and all } z \in B, \\ &\psi_0^{(j)}(\alpha_n z + \beta_n) \underset{B}{\rightrightarrows} f(z) - \varphi_0^{(j)}(\zeta) - \chi_{j,\nu_0}(\zeta) && \text{for } n \to \infty, \end{aligned}$$

which implies $\phi^{(j)}(\alpha_n z + \beta_n) \underset{B}{\rightrightarrows} f(z)$ for $n \to \infty$.

(b) Let us now assume that $\zeta \notin \partial G_\nu$ for all $\nu \in I$ or $\zeta = \infty$. Then there exist sequences $\{\nu_m\}$ with $\nu_m \in I$ and $\{\zeta_m\}$ with $\zeta_m \in \partial G_{\nu_m} \cap \mathbb{C}$ such that $\zeta_m \to \zeta$ for $m \to \infty$. We consider the polynomials χ_{j,ν_m} with $\chi_j(z) = \chi_{j,\nu_m}(z)$ if $z \in G_{\nu_m}$. According to the translation properties of $\psi_0^{(j)}$ there exist for each $m \in \mathbb{N}$ sequences $\{\alpha_n^{(m)}\}_{n\in\mathbb{N}}$ and $\{\beta_n^{(m)}\}_{n\in\mathbb{N}}$ with

$$\begin{aligned} &\alpha_n^{(m)} \to 0, \ \beta_n^{(m)} \to \zeta_m && \text{for } n \to \infty, \\ &\alpha_n^{(m)} z + \beta_n^{(m)} \in G_{\nu_m} && \text{for all } n \in \mathbb{N} \text{ and all } z \in B, \\ &\psi_0^{(j)}(\alpha_n^{(m)} z + \beta_n^{(m)}) \underset{B}{\rightrightarrows} f(z) - \varphi_0^{(j)}(\zeta_m) - \chi_{j,\nu_m}(\zeta_m) && \text{for } n \to \infty. \end{aligned}$$

For fixed $n \in \mathbb{N}$ we choose an index $n_m > m$, such that $a_m := \alpha_{n_m}^{(m)}$ and $b_m := \beta_{n_m}^{(m)}$ satisfy the following properties simultaneously:

$$|a_m| < \frac{1}{m}, \qquad |b_m - \zeta_m| < \frac{1}{m},$$

$$\max_B |\varphi_0^{(j)}(a_m z + b_m) - \varphi_0^{(j)}(\zeta_m)| < \frac{1}{m},$$

$$\max_B |\chi_{j,\, \nu_m}(a_m z + b_m) - \chi_{j,\, \nu_m}(\zeta_m)| < \frac{1}{m},$$

$$\max_B |\psi_0^{(j)}(a_m z + b_m) - f(z) + \varphi_0^{(j)}(\zeta_m) + \chi_{j,\, \nu_m}(\zeta_m)| < \frac{1}{m}.$$

We have $a_m z + b_m \in \mathcal{O}$ for all $m \in \mathbb{N}$ and letting $m \to \infty$, we obtain

$$a_m \to 0, \qquad b_m \to \zeta,$$

$$\phi^{(j)}(a_m z + b_m) \underset{B}{\Longrightarrow} f(z).$$

This proves assertion (E) and finishes the proof of Theorem 1.

5. UNIVERSAL FUNCTIONS IN THE SPACE OF HOLOMORPHIC FUNCTIONS

We denote by $U(\mathcal{O})$ the set of all multiply universal functions having the properties (A–E) of Theorem 1, and we deal with the question of whether it might be considered a normal or nonnormal feature for a function to belong to $U(\mathcal{O})$.

With the notations in the proof of Theorem 1 we consider the sequence

$$\mathcal{O}_n = \bigcup_{\nu \in I_n} G_{\nu,\, n}$$

of exhausting open sets for the open set $\mathcal{O}$. For a function $F \in H(\mathcal{O})$ and $n \in \mathbb{N}$ we set

$$d_n(F) := \max_{\overline{\mathcal{O}_n}} |F(z)|; \qquad d(F) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(F)}{1 + d_n(F)}.$$

If $F_1, F_2 \in H(\mathcal{O})$ we define their distance by

$$d(F_1, F_2) := d(F_1 - F_2).$$

The set $H(\mathcal{O})$ established with this metric is a complete metric space, and for a sequence $\{F_k\}$ of functions $F_k \in H(\mathcal{O})$ we have

$$\lim_{k \to \infty} d(F_k, F) = 0 \qquad \text{if and only if} \quad F_k(z) \underset{\mathcal{O}}{\Longrightarrow} F(z).$$

This shows that d is a "natural" metric (which is induced by the compact convergence) in $H(\mathcal{O})$.

We now compare the subset $U(\mathcal{O})$ with $H(\mathcal{O})$ and prove the following result.

THEOREM 2. *Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$, be an open set with simply connected components. Then $U(\mathcal{O})$ is dense in $H(\mathcal{O})$.*

Proof.

1. By Theorem 1 we have $U(\mathcal{O}) \neq \emptyset$. Let ϕ be any function in $U(\mathcal{O})$.

(a) Suppose that $\gamma \neq 0$ is any constant; then it is clear that the function $\gamma\phi$ also belongs to $U(\mathcal{O})$.

(b) Suppose that P is any polynomial. We shall prove that the function $\phi_P := \phi + P$ also belongs to $U(\mathcal{O})$. It is easy to see that ϕ_P satisfies the properties (A), (Da), and (Db) in Theorem 1; (B) and (C) follow from [19, Theorem 3] and [20, Theorem] respectively. Therefore we only have to verify that ϕ_P satisfies (E).

To this end let any $\zeta \in \partial\mathcal{O}$, any compact set $B \in \mathcal{M}$, any $f \in A(B)$, and $j \in \mathbb{Z}$ be given. We assume also that $\phi_P^{(j)}$ is the derivative of order j if $j \in \mathbb{N}_0$ or an (arbitrary but fixed) antiderivative of order $|j|$ if $-j \in \mathbb{N}$. If $-j \in \mathbb{N}$ we further assume that $P^{(j)}$ is a fixed antiderivative of order $|j|$ for the polynomial P on $\mathbb{C}$ (and hence on $\mathcal{O}$), and we may choose the antiderivative $\phi^{(j)}$ for ϕ on $\mathcal{O}$, so that we have

$$\phi_P^{(j)}(z) = \phi^{(j)}(z) + P^{(j)}(z) \qquad \text{for} \quad z \in \mathcal{O}.$$

(For $j \in \mathbb{N}_0$ this identity holds trivially.)

We first suppose that $\zeta \neq \infty$. Since the function $\phi^{(j)}$ satisfies property (E) in Theorem 1, there are sequences $\{a_n\}$ and $\{b_n\}$ with $a_n \to 0$, $b_n \to \zeta$ for $n \to \infty$, such that $a_n z + b_n \in \mathcal{O}$ for all $z \in B$, $n \in \mathbb{N}$, and

$$\phi^{(j)}(a_n z + b_n) \underset{B}{\Longrightarrow} f(z) - P^{(j)}(\zeta) \qquad (n \to \infty),$$

which implies $\phi_P^{(j)}(a_n z + b_n) \underset{B}{\Longrightarrow} f(z)$.

Let us now suppose that $\zeta = \infty$. Then we can choose a sequence $\{\zeta_m\}$ with $\zeta_m \in \partial\mathcal{O} \cap \mathbb{C}$ and $\zeta_m \to \infty$ for $m \to \infty$. For any $m \in \mathbb{N}$ there are sequences

$\{a_n^{(m)}\}_{n\in\mathbb{N}}$ and $\{b_n^{(m)}\}_{n\in\mathbb{N}}$ with $a_n^{(m)}\to 0$, $b_n^{(m)}\to\zeta_m$ for $n\to\infty$ such that $a_n^{(m)}z+b_n^{(m)}\in\mathcal{O}$ for all $z\in B$, $n\in\mathbb{N}$, and

$$\phi^{(j)}(a_n^{(m)}z+b_n^{(m)}) \underset{B}{\Longrightarrow} f(z)-P^{(j)}(\zeta_m) \qquad (n\to\infty).$$

For any $m\in\mathbb{N}$ there is an integer $n_m>m$ such that $\alpha_m:=a_{n_m}^{(m)}$, $\beta_m:=b_{n_m}^{(m)}$ satisfy the following conditions simultaneously:

$$|\alpha_n|<\frac{1}{m}, \qquad |\beta_m-\zeta_m|<\frac{1}{m},$$

$$\max_B |P^{(j)}(\alpha_m z+\beta_m)-P^{(j)}(\zeta_m)|<\frac{1}{m},$$

$$\max_B |\phi^{(j)}(\alpha_m z+\beta_m)-f(z)+P^{(j)}(\zeta_m)|<\frac{1}{m}.$$

It follows that $\alpha_m\to 0$, $\beta_m\to\zeta=\infty$ for $m\to\infty$ and

$$\phi_P^{(j)}(\alpha_m z+\beta_m) \underset{B}{\Longrightarrow} f(z).$$

This shows that $\phi_P^{(j)}\in U(\mathcal{O})$.

2. To prove Theorem 2, we have to show: Given any function $F\in H(\mathcal{O})$ and any $\varepsilon>0$ then there exists a universal function $\phi\in U(\mathcal{O})$ with $d(\phi, F)<\varepsilon$.

If we take an arbitrary function $\phi_0\in U(\mathcal{O})$, then we have $\lim_{t\to 0} d(t\phi_0) = 0$. Hence we can choose a constant $\gamma>0$ so that $d(\gamma\phi_0)<\varepsilon/2$.

According to Runge's approximation theorem there exists a sequence $\{P_k\}$ of polynomials with $P_k(z) \underset{\mathcal{O}}{\Longrightarrow} F(z)$. Therefore we can find a polynomial P with $d(P, F)<\varepsilon/2$. By Step 1 the function

$$\phi(z):=\gamma\phi_0(z)+P(z)$$

belongs to $U(\mathcal{O})$ and satisfies

$$d(\phi, F)=d(\gamma\phi_0+P, F)\leqslant d(\gamma\phi_0)+d(P, F)<\varepsilon,$$

which proves Theorem 2.

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